The Chaotic Hierarchy

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The complexity of dynamical behavior possible in nonlinear (for example, electronic) systems depends only on the number of state variables involved. Single-variable dissipative dynamical systems (like the single-transistor flip-flop) can only possess point attractors. Two-variable systems (like an LC-oscillator) can possess a one-dimensional attractor (limit cycle). Three-variable systems admit two even more complicated types of behavior: a toroidal attractor (of doughnut shape) and a chaotic attractor (which looks like an infinitely often folded sheet). The latter is easier to obtain. In four variables, we analogously have the hyper-toroidal and the hyper-chaotic attractor, respectively; and so forth. In every higher-dimensional case, all of the lower forms are also possible as well as "mixed cases" (like a combined hypertoroidal and chaotic motion, for example). Ten simple ordinary differential equations, most of them easy to implement electronically, are presented to illustrate the hierarchical tree. A second tree, in which one more dimension is needed for every type, is called the weak hierarchy because the chaotic regimes contained cannot be detected physically and numerically. The relationship between the two hierarchies is posed as an open question. It may be approached empirically – using electronic systems, for example.

1. Introduction

Electronics, with its curious blend of linear and nonlinear elements, provides the second major field of application for dynamical systems theory, after mechanics. (The third is formed by chemical reaction systems and their analogues including biophysical and ecological systems.) Aside from linear (for example, phase-shifting) elements, there are both nearly linear amplifiers and strongly nonlinear "switching devices". (In addition, there is of course the neon tube, described by partial differential equations; the cathod ray tube; etc.) Electronic switching systems can be described by a combination of more or less linear "slow variables" and some strongly nonlinear "fast" variables of the singular-perturbation type. For purposes of simplification, these fast variables are frequently replaced by algebraic constraints (see, for example, [1, 2]).

Despite their ubiquity and ease of design, electronic systems have rarely been made use of so far in experimental dynamics. Reproducible performance, so important in the design of present-day computers, forms one extremum in a continuum of possibilities. At the other end stands a multitude of deterministic chaotic motions. Their classification still constitutes a major dynamical problem.

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Historically, the capability of electronic systems described by singular-perturbation type ordinary differential equations to produce nontrivial dynamical behavior was first seen by Cartwright and Littlewood [3]. The abstract dynamical theory of "horseshoe maps" (Smale [4]) was triggered by this work (or rather a sequel to it [5]). The early papers (by Cartwright and Littlewood and their followers) all dealt with non-autonomous – periodically forced – systems. The first paper in which an autonomous electronic system of singular-perturbation type was investigated with a view to complicated dynamical behavior, appeared much later (Smale [6]). While the 3-variable equation considered there is in principle capable of showing complicated motions, such trajectories were not discovered at first. Then in 1976, the chaos-generating power of this class of systems was independently described by Takens [7] and Rössler [8]. Takens [7] realized that a flow similar to one described earlier by Lorenz [9] is possible in such systems; Rössler [8] at first overlooked this possibility but instead found a new type of chaotic flow, the "three-dimensional blender", which is governed by a simpler, walking-stick shaped (horseshoe-like) cross-section. These results were soon reproduced and extended by Plant [10], Rabinovich [11], and Mira [12].

In the following, first the simplest type of chaotic flow that is possible in three dimensions (and in the present class of systems) will be outlined. Then the

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underlying principle will be exposed which permits a straightforward generalization to flows of maximal complexity in arbitrary dimensions. An application to symmetry-constrained systems will lead to the prediction of a new "scaling law". Finally, two prototypical "maps" will be considered which generate "weak" (as opposed to the former "strong") higher chaos.

2. The Rotating Taffy Puller

To understand the nature of chaos, one may either read Anaxagoras (456 B.C. [13]) or study the following mechanical mixing system which captures the essence of his model:

Two synchronously moving pairs of arms in Fig. 1a serve to automatically stretch and fold a piece of taffy – that sticky, expansible, sweet material from which caramel candy is made. Figure 1b shows a number of subsequent "shots" viewed from the side. Figure 1c shows the underlying law, and Fig. 1d an equivalent, simplified two-dimensional map. Figure 1e puts the taffy puller on a platform rotated in synchron with it ("lazy susan"), whereas Fig. 1f presents an essentially equivalent, but "infinitely thin" flow caused by an expending, rotating and ever again folded-over sheet of paper.

The pictures are self-explanatory. Stretching plus folding implies physical mixing, because neigh-

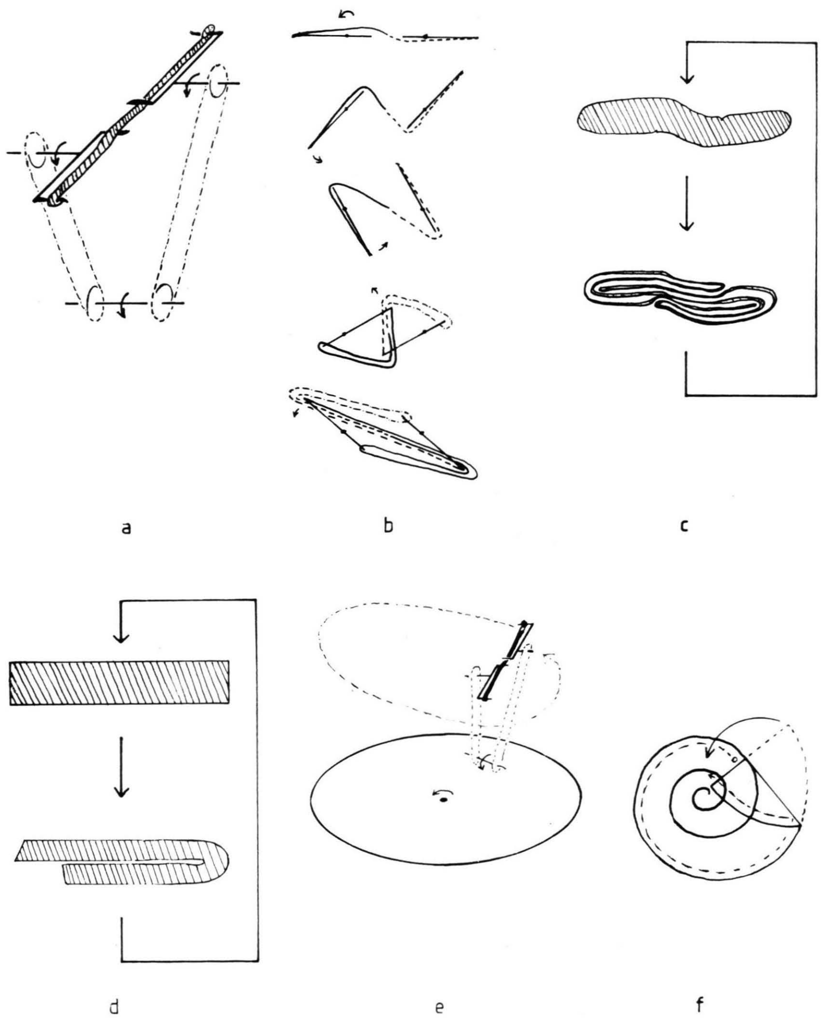

Fig. 1. a) A "taffy puller". b) Subsequent positions of the hands (over a half-cycle). c) The overall 2-dimensional map that would apply if the whole process were strictly 2-dimensional (think of two confining glass plates, one in front of and one behind the taffy). d) A simplified overall map. e) The rotating taffy puller. The dashed arrows in e and f each show the path in 3-space of a single point ("light bulb"). f) Zero-thickness version of a simplified rotating taffy puller ("paper model").

boring points (think of two raisins suspended in the taffy) diverge exponentially while previously distant points are brought close together. If the original piece of taffy consists of labelled ("black", or radioactive, respectively) molecules in the right-hand portion and non-labelled ones in the left-hand portion, it takes only about 24 rotations (or less than half a minute, if there is one rotation per second) until a "molecular sandwich" of black and white molecules has been achieved (if the original height is one centimeter and $(\frac{1}{2})^{24}$ is the width of a molecule).

The rotating taffy puller (Fig. 1e) forms a dynamical system in the sense of mathematics (cf. Smale [4]). If a tiny little light bulb is assumed to be present inside the taffy, and a dark room is assumed, then the visible light path, that is, the bulb's trajectory, forms an invertible flow (in the mathematical sense). This flow happens to be governed (in the simplest case) by the more or less walking-stick shaped two-dimensional diffeomorphism of Figure 1d. The "paper flow" of Fig. 1f again traces out such a line path, but with the additional simplifying assumption that the thickness of the original taffy is zero.

These pictures served to show that in three dimensions, it is very easy to generate an inextricably complicated "tangle" of hairlines ("spaghettis") that nevertheless is locally simple (everywhere parallel) and everywhere invertible and, moreover, governed by a very simple overall map. Such a disciplined tangle describes the simplest type of chaos.

It would be surprising if this simple spatial principle ("three-dimensional blender") would not readily arise spontaneously in very simple equations and systems – for example, electronic systems.

The principle of Fig. 1e was apparently known to Anaxagoras already. In his conservative cosmology, everything existed in a state of perfect mixture for an infinity of time. Then the sole immiscible substance, mind, at one point in space and time (nowadays called an initial condition) started a recurrent motion. In Greek, the technical term invented by Anaxagoras was "perichoresis" – "running around". Anaxagoras thus invoked the main ingredients of the above machinery to generate a process of "unmixing" – since he needed to explain the emergence of simple ingredients out of chaos. The machine of Fig. 1e was, of course, introduced by us with the opposite goal in mind – to explain the emergence of chaos out of simple things. Both views differ by the direction of time only.

The word chaos, incidentally, originally had the meaning of "emptiness" ("yawning") down to the time of Anaxagoras. Since the 4th century B.C., however, that is, since the time of Anaxagoras, its meaning has changed toward denoting mixture and turmoil.

3. An Equation

The paper flow of Fig. 1f if suggests that it should be very simple to set up an equation with the same behavior. Try the following: Take a two-variable linear oscillator with an unstable focus, like

$$\ddot{x} = -x + 0.15\,\dot{x} \tag{0}$$

or, equivalently,

$$\dot{x} = -y\,,$$
$$\dot{y} = x + 0.15\,y\,.$$

Then add a third variable (z) that tends toward a value close to zero whenever x is less than a certain threshold value, but rises autonomously as long as x is larger. A simple example is

$$\dot{z} = 0.2 + z\,(x - 10)\,,$$

where 10 is the threshold. What is still lacking is a folding mechanism. Up till now, as long as z is large, the "pivot" around which the linear suboscillator is rotating is just the same as before (uncoupled case). If the pivot is instead assumed displaced downwards along the y-axis during the elevation of z, however, the trajectory will when touching down again no longer reach the same point as if there had been no intervening elevation, but will be displaced somewhat downward and toward the left in proportion to its former height. This can be achieved by introducing a feedback from z to x – by subtracting z on the right-hand side of the first line, for example.

The result is the following three-variable equation [14]:

$$\dot{x} = -y - z\,, \quad \dot{y} = x + 0.15\,y\,,$$
$$\dot{z} = 0.2 + z\,(x - 10)\,. \tag{1}$$

A numerical simulation of this flow is presented in Figure 2. One sees that the equation indeed func-

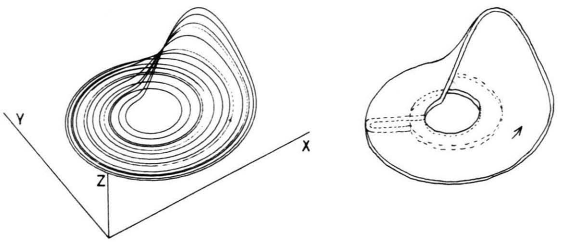

Fig. 2. The flow generated by (1) in 3-space. The same picture is shown twice (the second time schematically). Numerical simulation of (1), performed on an HP 9845B desk-top computer with peripherals using a standard Runge-Kutta-Merson integration routine. Initial conditions: $x(0) = 10$, $y(0) = 1$, $z(0) = 0$; $t_{\mathrm{end}} = 116$. Axes: $-20 \ldots 20$ for x, $-20 \ldots 15$ for y, $0 \ldots 30$ for z.

tions as expected. One also notices that an actual cross-section through the flow cannot be non-invertible – like an ideal paper flow is –, but must be invertible since the right-hand sides of (1) are all analytic functions, meaning that the existence and uniqueness theorems for ordinary differential equations (cf. [15]) are fulfilled. A cross-section through the flow of Fig. 2 therefore has roughly the form of the map shown in Fig. 1 d (walking-stick map).

The simplest algebraic equation generating such a map, incidentally, is

$$a_{n+1} = \gamma a_n (1 - a_n) - b_n\,,$$
$$b_{n+1} = (\delta b_n - \varepsilon)(1 - 2a_n)\,, \qquad (2)$$

which is easy to put into a computer. An appropriate set of parameters is $\gamma = 3.8$, $\delta = 0.4$, and $\varepsilon = 0.02$. A rectangular box that is never left under iteration is defined by the axis lengths $0.04 \ldots 0.98$ for a and $-0.0294 \ldots 0.0306$ for b, respectively (cf. [16], for pictures). Equation (2) is related to the well-known Hénon map [17], but contains an additional quadratic term ($2\delta a_n b_n$, second line) which is necessary if the "taffy" is indeed to be folded realistically. (Setting δ equal to zero in (2) – so that the latter becomes equivalent to an orientation-preserving version of Hénon's map –, incidentally also produces realistic foldings – but only if instead of the first iterate the second iterate is looked at. This second iterate is, however, again described by an equation with two quadratic terms on its right-hand side.) Therefore it is a matter of taste which of the two maps to choose for one's walking stick. While it is possible to study chaos purely on the basis of such maps (whereby many interesting results have already been found; cf. [18]), this implies a concentration on detail that interferes with the present purpose of classifying whole flows rather than their internal structure.

There is just one major "little detail" concerning folded-over maps that has to be mentioned here. Already when observing the taffy puller of Figure 1 a in a display window of a sweets shop (like the one in a shopping mall of Salt Lake City), one sees that at some places little non-expanding "pockets" form in the taffy. This occurs in the "knee" regions and is the consequence of a "lack of stretching" there. At least one "contracting pocket" almost always exists in the folded-over cross-section formed by a simple volume-contracting flow like that of Figure 2. Therefore, such flows almost always contain one or several periodic attractors – usually of very high periodicity – buried in the attracting chaotic regime (see [19] where this is rigorously shown for Hénon's map). The existence of these pockets nonetheless does not interfere with the fine structure of the overall attracting object (the "red line" with zero area but uncountably many times the length of the original rectangle [20]) that is formed as a limiting cross-section.

In other words, the flow of Fig. 2 and a rotating taffy puller (as in Fig. 1 e, but with a simplified folding mechanism) are very close indeed – if the fact that the area of a cross-section is not preserved by the realistic equation is taken into account. In terms of the original taffy picture, this means that the taffy must contain a liquid matrix that may be "squeezed out" in part during the stretching that precedes the folding over. If one wants to stick to the picture, this implies that the lost volume has to be made up for by new taffy. The latter is to be automatically poured in after every round in such a way that the original cross-section (think of a rectangular pan) is again filled out completely. The newly added material (which also serves the function of re-moisterizing the remnant of the last iteration) is conveniently pictured having a different color. If the original cross-section was red, for example, the new cross-section may be all white except for the walking-stick shaped red inset. At the next iterate, only the "walking-stick within the walking-stick" will be red, and so forth. Hence the attracting "red line" in the limit [41]. A computer with graphics can be very helpful in verifying that this picture, derived from dough-processing, indeed describes what occurs inside the map of (2), for example (cf. [16]).

4. A Singular-Perturbation Prototype

Equation (1) above is simple enough to provoke further investigation (see [21, 22, 23, 24]). On the other hand it is not close to any limiting equation whose solutions are known analytically. This drawback can be remedied by replacing the (single-threshold) third line in (1) by a bistable (double-threshold) element. As an example, consider the Eccles-Jordan [25] trigger which has the equations

$$\varepsilon \dot{u} = 1 + x - u - 2vu/(u+\varepsilon),$$
$$\varepsilon \dot{v} = 1 - v - 2uv/(v+\varepsilon)$$

with x the switching parameter. These equations generate, in the limit of ε going to zero, a perfectly letter-Z shaped algebraic function $u = f(x)$ in which the top and the bottom of the letter Z are stable while the slanted part is unstable [2]. Combining the Eccles-Jordan flip-flop with a negatively damped linear oscillator (of LC-type, for example) leads to an analytically tractable analogue to (1) – but of four variables. The mathematical prototype to this system, in turn, is one in which the third line of (1) has been replaced by a single-variable bistable element, like [20]

$$\varepsilon \dot{z} = (1-z^2)(x-1+z) - \delta z. \tag{1a}$$

Similar single-variable flip-flops are also known in electronics (cf. [1]). Note that z in (1a) again generates a (quadratic) hysteresis loop based on a letter Z (with the top and bottom at $+1$ and -1, respectively) if both ε and δ are allowed to approach zero and the parameter x is varied up and down starting from zero.

All one has to do, therefore, is string together solution pieces belonging to the two linear O.D.E.'s

$$\ddot{x} = -x + 0.15\,\dot{x} + 1$$

and

$$\ddot{x} = -x + 0.15\,\dot{x} - 1,$$

respectively.

Instead of doing this tedious work by hand, however, one can also have it done by a computer. A numerical simulation of the present equation, written in the form

$$\dot{x} = -y - 0.95\,z,$$
$$\dot{y} = x + 0.15\,y,$$
$$\varepsilon \dot{z} = (1-z^2)(x-1+z) - \delta z, \tag{1'}$$

is shown in Figure 3. Note that the former unit constant in front of z in the first line has been decreased somewhat (which is inessential), and that both ε and δ were chosen finite in the simulation. In the limit of both parameters going to zero, all the smooth "corners" in Fig. 3 become right angles, and the whole flow can be calculated analytically (including a one-dimensional cross-section through the flow) since the two subflows to be matched are linear [26, 20].

Mira [12] proposed a more complicated analogue to (1′) which has the asset that its cross-section assumes an especially simple, explicit form (as a one-dimensional map). Of course, when ε is chosen non-zero, no matter how small, suddenly a two-dimensional (rather than one-dimensional) map applies. An equation for a simple "imbedding diffeomorphism" that comes arbitrarily close to the actually applying two-dimensional map as ε approaches zero can also be indicated, however [20, 18].

Thus, the use of bistable elements like (1a), while not necessary for generating simple chaotic flows, nevertheless greatly facilitates analysis. At the same time, (1a) is very useful also when the design of new systems is at stake. For example, (1) was not found directly, but was derived in a number of steps from a singular-perturbation equation analogous to (1′) [8]. The most attractive design question, in the

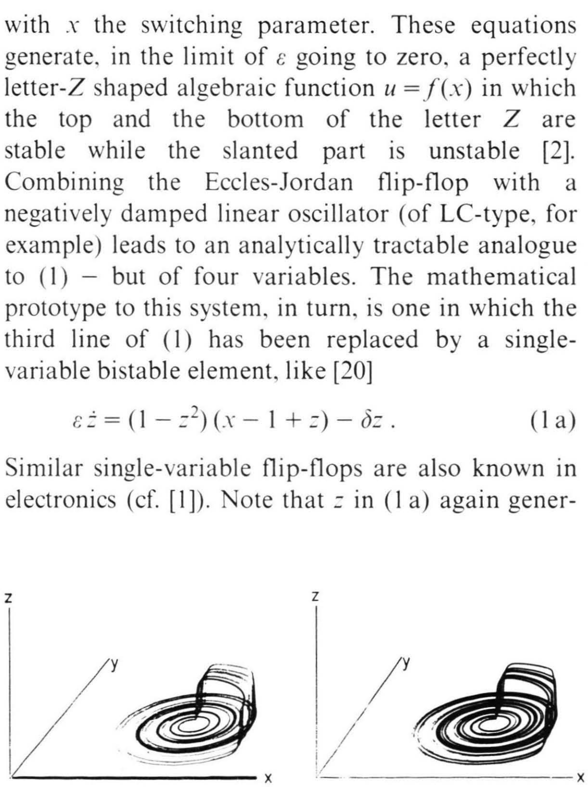

Fig. 3. Numerical simulation of (1′), a singular-perturbation analogue to (1). Simulation performed as in Figure 2. The three-dimensional flow is shown twice, the right-hand picture presented from a slightly different angle (parallel projections). A stereoscopic impression can be obtained by focusing one eye on each picture (either crossed or uncrossed); alternatively, a stereoscope may be used. Parameters: $\varepsilon = \delta = 0.03$. Initial conditions: $x(0) = 1$, $y(0) = 0$, $z(0) = -1$; $t_{end} = 200$. Axes: $-4\ldots 4$ for x and y, $0\ldots 8$ for z.

present context, is of course whether or not the simple principle that underlies the flow of Fig. 3 can be generalized to higher dimensions.

5. A "Ladder" Toward Higher and Higher Chaos

Equation (1′) produces the simplest form of "singular-perturbation chaos", as it may be called (cf. [10]). The limiting behavior of such an equation (for $\varepsilon \to 0$) can be obtained by geometrically "overlaying" two different two-dimensional flows (which in the present case are both linear). Many other interesting flows can be generated in the same way (see [26, 12, 11, 27, 28]). All of these systems produce "ordinary" (typically three-variable) chaos the simplest type of which was depicted in Figure 1. In every case, there is (i) a two-dimensional flow, (ii) a one-dimensional "escape border", (iii) another ("dashed") two-dimensional flow, and (iv) another one-dimensional "return threshold".

This four-step principle can be generalized. That is, the two flows can be made three-dimensional each, and the two borders two-dimensional. Then the limiting overall flow will be governed by a non-invertible map that is defined from a two-dimensional surface back onto itself. While the former one-dimensional surface could be elongated (and folded-over) in one direction only (as a "folded hairpin" map), the new two-dimensional map can in principle be elongated and folded over in two independent directions ("folded handkerchief" map). And just as the folded hairpin could be "blown up" toward becoming a two-dimensional invertible map (walking-stick diffeomorphism), so the folded handkerchief can be blown up toward becoming a three-dimensional invertible map (folded-towel diffeomorphism [16, 20]).

All one has to do in order to find such a flow is start out with an expanding three-dimensional linear flow. The simplest example has the form of a pointed, expanding screw. The equation of the linear subsystem then becomes

$$\begin{aligned}\dot{x} &= -y\,,\\ \dot{y} &= -x + 0.25\,y + w\,,\\ \dot{w} &= 0.05\,w\,,\end{aligned}$$

for example. To this flow, again a "threshold" (now in three-space) has to be added – for example, in the form of the third line of (1). All that then still remains to be done is insert an appropriate "feedback" from z back towards the other variables to make sure that the folding acquires the desired form.

A 4-variable equation which meets these constraints (as a direct analogue to (1)) is

$$\begin{aligned}\dot{x} &= -y - z\,,\\ \dot{y} &= x + 0.25\,y + w\,,\\ \dot{z} &= 2.2 + xz\,,\\ \dot{w} &= 0.05\,w - 0.5\,z\,. \qquad (3)\end{aligned}$$

An appropriate set of initial conditions is: $x(0) = -19$, $y(0) = z(0) = 0$, $w(0) = 15$ (see [16] for a stereoscopic simulation).

Figure 4 shows the relationship between this flow, drawn schematically (Fig. 4b), and the flow of Fig. 2 (Figure 4a). Note that now *two* mutually independent directions of stretching and folding-over are realized: one "across windings" within more or less the same height of the screw – such that more than one winding is covered by originally one winding (just as this is the case in Fig. 4a) –, and one "across turns" of the screw – such that more than one turn is covered by originally one turn. Indeed, Fröhling et al. [29], in a calculation of the Lyapunov characteristic exponents of (3) (obtained by numerically determining the three directions and rates of maximal divergence/convergence of neighboring trajectories [30]), found two positive such exponents, meaning that the attracting regime of (3) indeed contains two independent directions in which trajectories exponentially diverge from their neighbors.

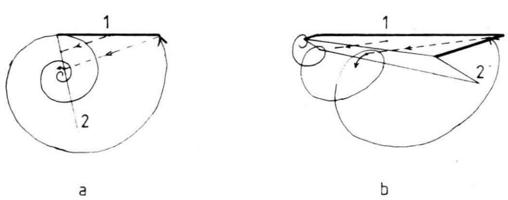

Fig. 4. a) Schematic representation of the "reinjection with overlap" that takes place between two essentially one-dimensional lines, in the flow of (1) and Figure 2. The two lines (1 and 2) are connected via an expanding 2D spiral. b) Analogous "reinjection with two directions of overlap" that takes place between two essentially two-dimensional planes, in the flow of (3). The two planes (1 and 2) are connected via an expanding 3D screw. Dashed lines = trajectories elevated into the third dimension (a) or the fourth dimension (b), respectively.

Once again it should be possible to obtain a singular-perturbation analogue to the non-singular equation presented. To this end, only the nonlinear third line of (3) has to be replaced by (1a) once more. Suitable parameter values for the resulting equation – (3′) – have yet to be identified, however.

As it happens, there exists another, more complicated 4-variable equation of the singular-perturbation type for which appropriate parameter values are already on hand. It has the asset that the limiting 2-dimensional cross-section (as ε approaches zero) is known analytically since the two limiting linear subflows are both especially simple (namely, concentric circles in one projection and straight lines through one point in the other [12]). This equation has the following form [31]:

$$\begin{aligned} \dot{x} &= w(-y-c) + (1-w)(z+c)\,, \\ \dot{y} &= w(x-1+a) \\ &\quad + (1-w)\,y(z+c)/(x+b-1)\,, \\ \dot{z} &= wz(-y-c)/(x-b) + (1-w)(a-x)\,, \\ \varepsilon\dot{w} &= w(1-w)(w-1+x) - \delta(w-\tfrac{1}{2})\,. \end{aligned} \tag{4}$$

A pertinent cross-section (at $x=0$, applying for $w<\frac{1}{2}$) has the following equation in the limit of both ε and δ approaching zero [31]:

$$\begin{aligned} y_{n+1} &= -c + \sqrt{2a-1+(2y_n+c)^2}\,, \\ z_{n+1} &= -2c + 2\sqrt{2a-1+(z_n+c)^2}\,. \end{aligned} \tag{5}$$

A numerical simulation of (4) is presented in Fig. 5a, whereby ε has been chosen rather small. (Increasing ε, δ to 0.005 while decreasing a to 0.48, yields an even "fuller" picture.) A numerical calculation of (5) for the same parameters as used in Fig. 5a (except ε, δ) is shown in Figure 5b. One sees that indeed an infinitely thin folded-towel (that is, folded-handkerchief) map applies in the limit. Each of the two subequations in (5) represents a folded-hairpin map – a "single-humped" map of the very type for which Li and Yorke [31a] originally proposed the term "chaos". Both sub-maps are identical (except for a change of coordinates; z corresponds to $y/2$) and uncoupled (in the limit).

It is possible to imbed the non-invertible map of (5) in a diffeomorphism (folded-towel type) by adding a third variable $\tilde{w}_{n+1} \neq$ const. (See [16], where this was done for the analogous case of two at first uncoupled logistic maps.) The deviation between this artificial diffeomorphism (in y, z, $\tilde{w}$), and the one that actually applies (in y, z, w) close to the limit of ε approaching zero, then becomes arbitrarily small as a corresponding parameter μ ($\propto \varepsilon^a$, $a>0$) in the y, z, $\tilde{w}$ map also approaches zero (cf. [18]).

In light of the examples considered up till now, it appears not too difficult to set up analogous, higher-dimensional equations ($n=5, 6, \ldots$) that likewise produce the most complicated type of

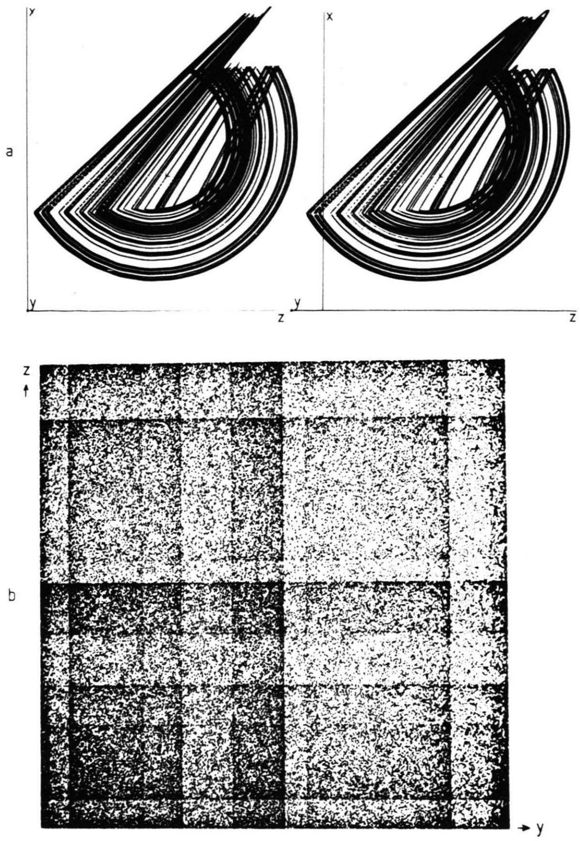

Fig. 5. a) Numerical simulation of (4), performed as in Figure 3. Stereoscopic representation of the x, y, z subflow. The dashed (more or less straight-appearing) trajectories correspond to values of w close to unity, the continuous (circular) ones to w close to zero. Parameters: $a=0.52$, $b=2$, $c=1.3$, $\varepsilon=\delta=0.001$. Initial conditions: $x(0)=0$, $y(0)=-1$, $z(0)=-1.5$, $w(0)=0.01$; $t_{\mathrm{end}}=460$. Axes: $-0.7\ldots1.4$ for x, $-3\ldots0$ for y, $-2.2\ldots-0.4$ for z. b) Numerical calculation of (5) for the same parameters and initial conditions as in a), except for $\varepsilon=\delta=0$. The multiply folded region (in the map) ranges from $y=-1.11$ to $y=-0.36$ and from $z=-2.21$ to $z=-0.73$. 120 000 iterates of the initial point are shown (at 11-digit accuracy).

chaos possible in the dimension in question. All one has to do is take an $n-1$ dimensional linear system with sufficiently many positive eigenvalues and combine it with a single bistable switching element. (The latter may afterwards be replaced by another nonlinear element containing but one quadratic term.) In every successful case, one obtains an $n-1$ dimensional diffeomorphism for a cross-section, with $n-2$ independent directions of exponential divergence of initial conditions. In the limit of ε approaching zero, there is a set of $n-2$ chaos-generating one-dimensional noninvertible maps that are either coupled or, as is the case with (5), uncoupled in the limit. While all these equations have yet to be formulated explicitly, there is no doubt that this can be done in principle (for example, with the aid of a specially designed computer program).

What has thus been shown is that there exists a hierarchy of simple ordinary differential equations in which every member implements "maximal chaos" with respect to its own dimension number.

6. Computer Turbulence

Digital computers are "nothing but" a subclass of singular-perturbation type ordinary differential equations in the simplest case (cf. [2]). The non perturbation-type ("slow") subsystems are in these systems set up in such a way that their behavior is non-oscillatory and stable. Thereby the feature most characteristic of (1′), (3′), and (4) is avoided (namely, that the slow variables are partly coupled and, moreover, unstable). In addition, there is on average one switching-type variable (like (1 a)) present for every slow variable (not counting the more or less slow additional variables that are formed by stray capacitances). The whole setup is such that the resulting overall system of ordinary differential equations remains "decomposable". This means that one can understand the whole system in a step-by-step fashion instead of having to take into account the dynamics of all variables simultaneously. This "building-block principle" is the reason that one can be so sure about the behavior of the overall system [2].

On the other hand, when in such a dynamical system just one little switch becomes slower than originally intended, and moreover in combination with some other (for example, stray) variable starts to form a subsystem of oscillatory type, the major preconditions for a chaotic mode of action are fulfilled. Aside from ordinary (typically three-variable) chaos with a single positive exponent of lateral divergence, higher forms of chaos are also possible. In fact their occurrence is facilitated once ordinary chaos has set in.

Thus, knowledge about the above systems (1′), (4) may help reduce the chances of certain types of "instability" occurring in digital computers. Some of these may be rather subtle (for example, input-dependent) and only give rise to irreproducible "hardware errors".

On the other hand, it appears legitimate to speculate whether knowledge of chaos could also be used positively – for example, in designing a futuristic computer that possesses a "probabilistic" rather than deterministic mode of action. The human biocomputer (the brain), which utilizes chaotic neurons (cf. [10]) and neuronal circuits (cf. [32]), possibly belongs to such a class of systems.

7. The Cartwright-Littlewood Hierarchy

The first people who saw chaos (although they did not name it so) in a dissipative dynamical system – in fact, an electronic system – were Cartwright and Littlewood [3]. These authors considered the periodically forced van der Pol oscillator,

$$\varepsilon\dot{x} = x - x^3 - y\,,$$
$$\dot{y} = x + a\cos bt\,, \tag{6}$$

and showed with analytical means that this system possesses, for certain windows in its parameters, an infinite (and even uncountable) number of "subharmonic" solutions of differing frequencies. A connection to Birkhoff's [33] analogous results on Hamiltonian systems (obtained with the help of Poincaré's [34] notion of a homoclinic point in a cross-section) was not drawn at first. This was done only later by Hayashi and Ueda [35, 36] in their numerical studies of another periodically forced two-variable nonlinear system.

In the light of the present examples, the system of (6) is not hard to understand. This is because it can be written as an autonomous 3-variable system in the following way:

$$\varepsilon\dot{x} = x - x^3 - r + 3\,, \tag{6'}$$
$$\dot{p} = -bq + \frac{p}{r}\left(x + a\frac{p}{r}\right), \quad \dot{q} = bp + \frac{q}{r}\left(x + a\frac{p}{r}\right),$$

where $r \equiv p^2 + q^2$, the squared amplitude of the rotation-symmetric p, q subsystem, has replaced y. (Note that $\dot{r} \equiv (p^2+q^2)^{\cdot} = \dot{y}$ by the chain rule, if $p(0) \neq 0$ and $q(0) = 0$.) The system of (6′) is equivalent to that of (6) for all values of y larger than -3, since r has been put equal to $y+3$, where 3 is an arbitrarily picked constant. It is easy to verify that the regime of interest in (6) does not contain any solutions tending toward $y < -3$. (6′) is therefore a valid substitute for (6). Looking at it as it stands, however, one sees also that (6′) is nothing but an "unnecessarily complicated" version of (1′). The two slow variables can be simplified progressively without losing chaos until an equation closely analogous to (1′) obtains.

This does not mean that interesting geometric arguments about the behavior of (6) in its own right could not also be made (see [37]). What is of interest in the present context, however, is that the Cartwright-Littlewood equation belongs into the very context of near-singular-perturbation chaos discussed above.

The principle which was used in the design of (6′) is universally applicable. It is, for example, possible to take a chaos-generating equation like (1) (or (6′)) and "split" one of its variables into two in the same way as this has been done in the transition from (6) to (6′). The new equation then still functions in the same way as before, with the squared amplitude of the split pair of variables playing the role of the original (unsplit) variable. On the other hand, there is now the possibility to introduce a new coupling, with parameter $a \neq 0$, from the new oscillatory component back to the rest of equation. An example based on (1) is

$$\dot{x} = -y - r + 50 + a\,p\,,$$

$$\dot{y} = x + 0.15\,y\,,$$

$$\dot{p} = -\omega q + (p/r)\,(0.2 + z\,(x-10))\,,$$

$$\dot{q} = \omega p\,(q/r)\,(0.2 + z\,(x-10))\,, \tag{7}$$

where $r \equiv p^2 + q^2 = z + 50$ and ω is the frequency of the new suboscillator. A Poincaré cross-section through this flow has for $a = 0$ (decoupled case) the form shown in Fig. 6 ("rotated walking-stick").

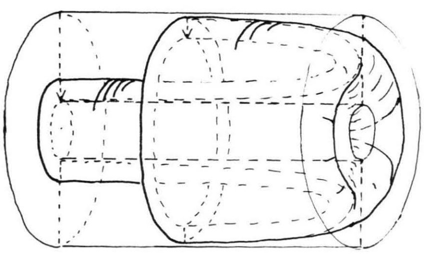

Fig. 6. Rotated walking-stick map. The outer cylinder (with inner bore) is the pre-image, the mushroom is the image. Such a map applies as a cross-section to the flow of (7) with $a = 0$ and $\omega \neq 0$ [39].

Another example, based on (6′), is

$$\varepsilon\,\dot{u} = -\frac{\omega}{\varepsilon}v + \frac{u}{r}(x - x^3 - y)\,,$$

$$\varepsilon\,\dot{v} = \frac{\omega}{\varepsilon}u + \frac{v}{r}(x - x^3 - y)\,,$$

$$\dot{p} = -q + p\,(x + a_1 p) + a_2\frac{u}{r}\,,$$

$$\dot{q} = p + q\,(x + a_1 p)\,, \tag{8}$$

where $y = p^2 + q^2 - 3$ as before; $x = u^2 + v^2 - C$ with C a sufficiently large positive constant; and $r = u^2 + v^2$. Note that the second and third lines of (6′) have been rendered here in somewhat simplified form. An appropriate new asymmetric "back-coupling" term has also been entered (with parameter a_2). Possible simplifications include dropping the r's and reducing the symmetry.

The present principle thus generates a new hierarchy. In (6′) with just one "splitting" present, a torus (with a "ring" map as a cross-section) obtains when $a = 0$. This ring can be distorted toward developing a "nose" (cf. [38]) that becomes progressively longer and eventually contains a walking-stick map, as the coupling constant a is made larger and larger. In (8) with its two "splittings", analogously a hypertorus (with a "thick-walled torus" map as a cross-section) obtains when $a_1 = a_2 = 0$. Its cross-section can be distorted toward possessing two "noses", one for each direction of rotation, that both grow in length until a full-fledged folded-towel map is formed [39], in the hypernose, as the two coupling constants are made larger and larger. "And so forth." The present hierarchy of "noses" is basically the same as the former; the underlying equations are just "unnecessarily complicated" as far as the generation of simple foldings is concerned.

8. "High" Chaos

"High higher chaos" is generated most conveniently by coupling together many identical subsystems that alone (or in pairs) are already capable of generating ordinary chaos. Examples are formed by many P.D.E.'s, but analogous discrete systems are also frequent. The simplest such system is a set of cross-inhibitorily coupled relaxation oscillators. If n such systems are arranged on a ring, the following equation applies for example:

$$\dot{x}_1 = 1.1 - x_1 - a(x_n + x_2 - 2x_1)\,, \qquad x_1 \bmod 1\,,$$
$$\dot{x}_2 = 1.1 - x_2 - a(x_1 + x_3 - 2x_2)\,, \qquad x_2 \bmod 1\,,$$
$$\vdots$$
$$\dot{x}_n = 1.1 - x_n - a(x_{n-1} + x_1 - 2x_n)\,, \quad x_n \bmod 1\,. \quad (9)$$

The individual suboscillators (x_i with $a = 0$) are idealized single-sweep relaxation oscillators each in which a modulo-type switching has replaced the fast second variable. Negative values of the x_i are admissible. As the coupling parameter a is increased symmetrically, the system eventually produces a folded "hyper-handkerchief" containing $n-1$ directions of strong stretching and folding over ("boiling-type turbulence" [40]).

What is especially interesting about such symmetric systems is that the pertinent map is symmetric too – reaching the "threshold of chaos" in all directions simultaneously. Thus, higher chaos is in the present case not reached in a stepwise manner as an underlying parameter is varied (first ordinary chaos, then first-level higher chaos, and so on), as this was the case in all of the previous systems, but rather emerges "directly". The process in which this occurs is a higher-order analogue to the better-known emergence of ordinary chaos under an increase of folding. (In (1), for example, such an increase can be achieved by letting the "undamping" parameter 0.15 rise starting from zero – as in the computer movie described in [41].) For this transition, Feigenbaum [42] showed that it observes a "scaling law" as far as the successive, ever closer-spaced appearance of higher and higher periodic solutions (ending in chaos) is concerned. In the present, more complicated case, an analogous law can be expected to hold true. Moreover, it appears possible that some of the phase transitions in physics that are known to obey a scaling law in space (cf. [43]) have properties in common with the present deterministic spatio-temporal phenomenon.

9. Weak Higher Chaos

So far, only clear-cut cases of higher chaos were considered. The stretching and folding-over was marked for every direction in which it occurred. A result of Newhouse et al. [44] shows that this is not necessarily so.

These authors showed that if one starts out with 3 uncoupled periodic motions in a dissipative system (as is the case in (8) for $a_1 = a_2 = 0$), introducing the slightest finite amount of coupling between the three independent suboscillators in general suffices for the formation of a chaotic attractor.

The mechanism underlying this result, as well as its meaning, can be appreciated best after considering a lower-dimensional analogue first. In 1962, Peixoto [45] investigated the analogous case of two uncoupled oscillators in a dissipative system (case of (6′) with $a = 0$). Here the formed quasiperiodic attractor on a torus is structurally unstable, giving in general way to a periodic attractor. The reason is that the nowhere closed flow on the torus (think of wrapping a thread around a doughnut so that the distance between two subsequent windings is always the same) is bound to become closed somewhere under almost any distortion. In general, both a repelling and attracting closed solution of arbitrarily high but finite periodicity are formed.

Analogously, an attracting hypertorus (like that formed in (8) with $a_1 = a_2 = 0$) is "structurally unstable" too. As it is distorted no matter how slightly, two formerly closely adjacent windings on a subtorus will again be shifted slightly out of their former relative position. This time, however, the shifting is not confined to one dimension as it was in the former case. The flat region in between the two windings can now be not only stretched or compressed (as before) but also folded over. Newhouse et al. [44] succeeded in showing that a combination of stretching and folding is bound to occur somewhere under almost any distortion. Thus, an arbitrarily small (though finite) analogue to the "large nose" that was necessary for squeezing chaos out of a two-torus ((6′) with a large) can be found in every neighborhood of the uncoupled case in (8).

This "miniaturization" principle can most certainly be generalized. By incorporating another "splitting" into (8), a 5-variable system with an attracting hyper^2 torus (T^4) is obtained. In this case, an arbitrarily small (though finite) analogue to the two independent large noses that were necessary in order to squeeze higher chaos out of a three-torus ((8) with both a_1 and a_2 large) can be expected to be found in every neighborhood of the uncoupled case; and so forth. This hierarchy-generating generalization has yet to be demonstrated formally. A precondition (ready occurrence of higher tori) was recently established by Sell [46].

The present hierarchy is "weak" because the resulting "near-quasiperiodic chaos" is physically inobservable by definition. Not surprisingly, there is also the possibility of "mixed" weak-strong chaos. An example is provided once more by (8). Here the cross-section of Fig. 6 applies when a_1 is sufficiently large to generate chaos in a subsystem (namely, (6′)) while a_2 is still zero.

In such a map, again an arbitrarily weak back-coupling ($a_2 \neq 0$) suffices for generating a new independent direction of stretching and folding over, however small. This can be seen most easily for the case where the frequency ω of the still uncoupled driving system is made a weak function of one of the variables of the driven system ($\omega = 1 + \varepsilon p$, for example). Any perturbation of symmetry in the last two lines of (8) is bound to have such an effect. The rotational fate of a point in Fig. 6 then suddenly depends on the point's coordinates within the walking-stick submap. This has the consequence that a very complicated "schlieren" structure develops in the direction of rotation, even though there is still

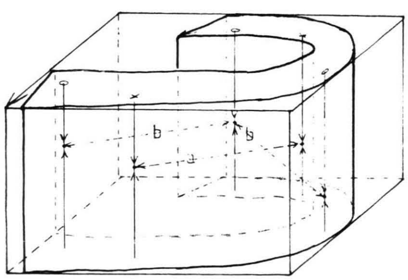

Fig. 7. Stacked walking-stick map. Two sets of periodic fixed points and their attracting manifolds have been entered schematically (a = period 2, b = period 3). Such a map applies as a cross-section to the flow of (10) with $a = 0$.

no back-coupling. The reason is that arbitrarily close pairs of points in the walking-stick map can belong to periodic solutions of differing periodicity (cf. [4]), and therefore can have a discretely different mean effect on the rotation rate, a difference that is bound to become arbitrarily large as time progresses. The result is a non-analytic "shearing" of the map into different "layers" (or schlieren). If now the slightest back-coupling (from the angular position, to the walking-stick) is introduced, somewhere a new stretching and folding over (roughly in the direction of rotation) has to occur, no matter how minute.

Unexpectedly, the same result applies to an even simpler class of equations already. Any passively driven "phase-shift variable" (an RC element, for example) added to a chaotic system also generates a schlieren effect. An example is the following system:

$$\begin{aligned} \dot{x} &= -y + aw, \\ \dot{y} &= -x + 0.15\,y, \\ \dot{z} &= 0.2 + z(x - 10), \\ \dot{w} &= x - by. \end{aligned} \tag{10}$$

Here (1) has been combined with a linear, trivially following variable w. If the back-coupling parameter a is zero, the map of Fig. 7 applies.

Every periodic sequence of driving states, having its own mean, generates a set of discretely "mean steady state values" (periodic fixed points) in the map. This results in a non-analytic vertical "schlieren structure" this time. It is bound to have the same consequence as before once a in (10) is given a very small nonzero value.

The significance of weak chaos constitutes an open problem. There are two possibilities: a) it is a normal precursor to strong chaos; b) it is typical for near-zero couplings and in general disappears soon thereafter, thus lacking connection to the strong chaos that may (or may not) develop under much stronger coupling. (See also below.)

10. The Dimensionality of Attractors Problem

Nonlinear dynamical systems of a realistic kind (like electronic systems) are in general dissipative (divergence > 0 on the average). Moreover, their solutions are in general bounded by physical reasons. Both constraints taken together imply that these systems have to possess at least one attractor. Attractors are always of measure zero in state space since all volumes between different initial conditions (think of a little cube) go to zero as t goes to infinity in a dissipative O.D.E. Therefore, any attractor can at most have $n-1$ dimensions.

The simplest attractors are zero-dimensional (points). They are maximal ($n-1$ dimensional) only in one-dimensional systems. In two-dimensional systems, the maximal attractor is a closed 1-dimensional curve (periodic attractor). In 3 dimensions, there are two different two-dimensional objects possible: an attracting torus (closed curve in a cross-section) and an attracting infinitely often folded sheet ("red line" in a cross-section).

In 4 dimensions, three types of 3-dimensional attractors are possible: an attracting hypertorus; an attracting higher-chaotic regime (as in (3) and (4)); and an attracting mixed chaotic and toroidal regime (as in (7) and (8)). These cases can be distinguished in terms of the pattern of their Lyapunov characteristic exponents (cf. [47]). The sign patterns ("signatures") corresponding to the three cases are $(0\ 0\ 0\ -)$, $(+\ +\ 0\ -)$, and $(+\ 0\ 0\ -)$, respectively. Hereby only the first two exponents make a difference (because one "zero" is always present in a recurrent flow – corresponding to the neutral forward direction –, and one "minus" is needed for the attraction). An even greater diversity applies in higher dimensions ($n > 4$) since the $n-2$ zeros and/or plusses can be divided up in even more ways.

Thus, the number of plusses always reflects the order of chaos shown by an attractor. If the chaos is weak in one dimension, the corresponding plus is replaced by a (numerically undetectable) $+\varepsilon$. The conjectured generalized N.T.R. [44] theorem of the previous Section suggests that whenever there are m apparent zeros found, one is genuine, one is a $-\varepsilon$, and the rest $(m-2)$ are $+\varepsilon$'s. The topological dimensionality of an attractor is therefore in general equal to the sum of its plusses and near-zeros, minus one.

Two more remarks are on line. a) The topological dimensionality of an attractor is a lower bound to its "fractal dimensionality" [48]. The latter can be calculated too from the Lyapunov spectrum [49]. As it turns out, however, more than one such number is needed in order to unequivocally characterize attractor dimensionality in the case of higher chaos [50]. b) The order of chaos of an attractor may be its most important characteristic. Experimentation on simple systems (like (3), (4), (7), (8)) may reveal that apparent zeros are rather rare in parameter space (cf. hypothesis b) if Section 9). In this case, only a plus could in general be counted on when the spanning of an additional attractor dimension is at stake.

11. Discussion

Electronics is a "synthetic" science. It serves mainly not to explain natural phenomena, but to create them. It seemed therefore a good strategy to present for once not theorems about nature (especially turbulence), nor theorems about chaos-generating maps, but recipes as how to generate interesting types of new behavior by electronic means – that is, by means of ordinary differential equations that are easy to implement electronically.

In this way, the outline of an emerging hierarchy of complicated dynamical phenomena could be presented. So far, there have been only few papers on the dimensionality of attractors. An early qualitative paper, which however does not deal with attractors proper (but with Smale-type "basic sets" which correspond to chaotic separatrices, so to speak), is by Franks [51]. More recently, the numerical work of Shimada and Nagashima [52], Packard et al. [53] and Fröhling et al. [29] has generated an awareness that the hierarchy problem has practical relevance.

Studies on the spectrum of Lyapunov characteristic exponents in coupled nonlinear systems (O.D.E.'s, P.D.E.'s and F.D.E.'s) are currently gaining interest; see, for example, [54, 55, 56]. The main proposal made above appears to be new: to discard the near-zero positive characteristic exponents in defining the "relevant dimensionality" of a chaotic regime. Everything becomes very simple this way. One plus means "minimal" (essentially 3-variable) chaos while, at the other extremum, $n-2$ plusses define "maximal" (fully n-variable) chaos.

This proposal is part of a larger philosophy. Possibly (and even probably) none of the systems described above possesses even a single positive Lyapunov characteristic exponent for the parameters assumed – if infinite computational accuracy as well as infinite computation length is presupposed as theoretically required. The reason is the likely formation of minute contracting pockets inside most folded maps (see Section 3). The fact that the theoretically existing imbedded periodic attractors do not exist empirically for most parameters (cf., for example, [21, 22]) suggests the following theoretical move: to conceptually replace the equation at hand by a close analogue (for example, the same equation at a close by point in parameter space) which happens to be "clean". This suggestion to allow a minimal idealization (cf. also [57]) could have been avoided by replacing the above examples (which all generate the easiest to obtain – folded – type of chaos and higher chaos) by more complicated equations showing the "cut" type of chaos [9, 41] and higher chaos [20] – which can be kept free from polluting periodic attractors. This would have solved only half the problem, however. The second type of "impurity" described above (directions of weak chaos, generated in equally small "expanding pockets") would have called for essentially the same move once more.

Disregarding both types of "pollution" brings a simple order into the chaotic world. This order may be artificial in the sense of holding true only for more complicated types of equations that are much rarer than those governing the impure cases. Whether or not nature actually tends to "gloss over" impurities – as quantum mechanics suggests – is presently open.

At this point, a more basic problem may be raised. If chaos is such a major new phenomenon in three dimensions as was suggested with Fig. 1: is it possible that something similarly new, but completely unrelated to chaos, may analogously emerge in 4 dimensions, and so forth? If so, there would exist in dynamical systems not only a "chaotic hierarchy", but also a more general one in which chaos would occupy just one level even with all its higher-dimensional generalizations taken into account. The situation would then be comparable to the existing separation between chaos on the one hand, and toroidal motions (where knowing one in a sense means knowing all) on the other. A recent topological finding (existence of a locally smooth, but nonetheless "non-smoothable" 4-dimensional manifold called "fake R^4" [58]) also encourages search for new dynamical phenomena in higher dimensions.

The quest for what is "beyond chaos" is interesting for an additional reason. Fluids containing chemical reaction mixtures certainly are chaotic on a molecular level (Hamiltonian chaos). Yet the same systems produce radically different (dissipative) phenomena on a higher level at the same time. It thus is in principle possible to proceed from chaos to a more regular regime without giving up chaos. No more than 4 variables appear to be needed [59]. This ready emergence of "order" out of chaos also points to the possibility of a "second hierarchy".

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